University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 10

Due 11:59pm Thursday, March 20, 2025

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1. **When is the QCD interaction attractive?**

   Write the amplitude for *tree-level* scattering of a quark and antiquark of different flavors (say u and $\bar{d}$) in the t-channel (in Feynman $\xi = 1$ gauge). Compare to the expression for $e\bar{\mu}$ scattering in QED.

   First fix the initial colors of the quarks to be different – say the incoming u is red and the incoming $\bar{d}$ is anti-green. Show that the potential is repulsive.

   Now fix the initial colors to be opposite – say the incoming u is red and the incoming $\bar{d}$ is anti-red – so that they may form a color singlet. Show that the potential is attractive.

   Alternatively or in addition, describe these results in a more gauge invariant way, by characterizing the potential in the color-singlet and color-octet channels.

   You can do this problem either by choosing a specific basis for the generators of $\mathsf{SU}(3)$ in the fundamental (a common one is the Gell-Mann matrices), or using more abstract group theory methods.

2. **Spinors in other dimensions.** The remaining two problems take place in $2+1$ and $1+1$ dimensions respectively and involve Dirac spinors. The following will be useful.

   (a) Show that for both $D = 2$ or $D = 3$ we can find a *two* dimensional representation of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Hint: use the Pauli matrices with appropriate some factors of $\mathbf{i}$.

   (b) What is $\mathrm{tr}\gamma^\mu\gamma^\nu$ in these two cases?

   (c) What is $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho$ in $D = 2$ and $D = 3$ respectively?

   (d) What is $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$?

3. **Where to find a Chern-Simons term.**

   Consider a field theory in $D = 2 + 1$ of a massive Dirac fermion, coupled to a *background* $\mathsf{U}(1)$ gauge field A with action:

   $$S[\psi, A] = \int d^3x \bar{\psi} \left(\mathbf{i}\not{D} - m \right) \psi$$

   where $D_\mu = \partial_\mu - \mathbf{i}A_\mu$. A_μ is not dynamical for the purposes of this problem. Use the two-component spinors from the previous problem.

(a) Convince yourself that the mass term for the Dirac fermion in $D = 2 + 1$ breaks parity symmetry. By parity symmetry I mean a transformation $\psi(x) \to \Gamma\psi(Ox)$ where $\det O = -1$, and Γ is a matrix acting on the spin indices, chosen so that this operation preserves $\bar{\psi}\partial\!\!\!/\psi$.

(b) We would like to study the effective action for the gauge field that results from integrating out the fermion field

$$e^{-S_{eff}[A]} = \int [D\psi D\bar{\psi}] e^{-S[\psi,A]}.$$

Focus on the term quadratic in A:

$$S_{eff}[A] = \frac{1}{2}\int đ^D q A_\mu(q)\Pi^{\mu\nu}(q)A_\nu(q) + ...$$

We can compute $\Pi^{\mu\nu}$ by Feynman diagrams. Convince yourself that Π comes from a single loop of ψ with two A insertions.

(c) Evaluate this diagram using dim reg near $D = 3$. Show that, in the low-energy limit $q \ll m$ (where we can't make on-shell fermions),

$$\Pi^{\mu\nu} = a\frac{m}{|m|}\epsilon^{\mu\nu\rho}q_\rho + ...$$

for some constant a. Find a. Convince yourself that in position space this is a Chern-Simons term, $S[A] = \frac{k}{4\pi}\int A \wedge dA$. Find the level k.

(d) Redo this calculation by doing the Gaussian path integral over ψ.

4. **A bit about the Schwinger model.**

Consider a version of QED in $D = 1 + 1$:

$$S[A,\psi] = \int d^2x \left(\sum_{\alpha=1}^{N_f} \bar{\psi}_\alpha \left(\mathbf{i}D\!\!\!\!/ - m_\alpha\right)\psi_\alpha - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} \right) \quad (1)$$

where each ψ_α is a 2-component spinor, and $D\!\!\!\!/ = \gamma^\mu(\partial_\mu + \mathbf{i}A_\mu)$. Use the two-component spinors from the problem above.

(a) What are the mass dimensions of the coupling g?

The model with $N_f = 1$ was solved by Schwinger in 1962. It is a model of confinement of charge, essentially because, even classically, the Coulomb potential grows linearly in $D = 1 + 1$, just like a confining potential. When $\theta = \pi$, there is a line of first-order phase transitions as a function of m_1, terminating at a critical point at $m_1 = m_{\rm cr} \approx 0.3335g$. (When $\theta = \pi$ things are less interesting.) We'll set $\theta = \pi$ for the whole problem.

If m_2 is very heavy, the model should reduce the $N_f = 1$ model at very low energies, and so should still have a critical point at some value of m_1. However, we can expect that the critical value of m_1 will be modified by some dependence on m_2 that goes away when $m_2 \to \infty$.

Predict the shape of the critical curve $m_{1\rm cr}(m_2)$ in the regime $m_2 \gg g$.

If I were feeling mean I would end the question here. Since I am feeling friendly, I will say a few more words. The heavy fermion with mass m_2 will contribute to the vacuum polarization for the gauge field.

(b) Compute the contribution to the vacuum polarization of the gauge field from the heavy fermion field $\Pi^{\mu\nu}(q)$ in the limit $m_2 \gg g$, as a function of m_2 and g, for $q^2 \ll m_2^2$.

(c) Interpret your previous result as a correction to g, i.e. to the coefficient $\frac{1}{g_{\rm eff}^2}$ of the Maxwell term.

(d) Using the known formula for the critical value of m in the $N_f = 1$ case, predict the critical curve as a function of m_2.

I got this problem from a talk I saw today at the KITP.