

Physics 215A QFT Fall 2023 Assignment 3

Due 12:30am Thursday, October 16, 2026

1. **Classical Maxwell theory.** [Peskin problem 2.1, lightly edited] Classical electromagnetism follows from the action

$$S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right), \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

- (a) Derive two of the Maxwell equations (Gauss' Law and Faraday's Law) as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables:

$$0 = \frac{\delta S[A]}{\delta A_\mu(x)}.$$

Write the equations in the standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

- (b) Construct the energy-momentum tensor for this theory, when $j^\mu = 0$. Note that the usual procedure

$$T_\nu^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta_\nu^\mu$$

does not result in a symmetric tensor. (It is also not gauge invariant.) To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} \equiv T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu,$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2} (E^2 + B^2), \quad \vec{S} = \vec{E} \times \vec{B}.$$

- (c) [Bonus problem] A better way to think about the energy-momentum tensor is to regard it as the response to a change in the background metric. (This is why it appears as a source in Einstein's equations, which is the equation of motion for the metric field.) To couple the Maxwell theory to a general background metric $g_{\mu\nu}$, we replace all the $\eta_{\mu\nu}$ s with $g_{\mu\nu}$ s:

$$S[A, g] = \int d^4x \sqrt{g} \left(-\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} + j^\mu A_\mu \right)$$

where the factor of $\sqrt{g} \equiv \sqrt{|\det g|}$ is required to make the integration measure coordinate-invariant, and $g^{\mu\nu}$ is the inverse metric: $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$. Compare the resulting energy-momentum tensor

$$T_g^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S[A, g]}{\delta g_{\mu\nu}} \Big|_{g_{\mu\nu}=\eta_{\mu\nu}}.$$

with that of the previous part.

Notice that $T_g^{\mu\nu}$ is automatically symmetric and gauge invariant.

[Some useful identities are:

$$\frac{\delta g^{\mu\nu}(x)}{\delta g_{\rho\sigma}(y)} = -g^{\mu\rho} g^{\nu\sigma} \delta^D(x - y) \text{ and } \frac{\delta \det g(x)}{\delta g_{\mu\nu}(y)} = \delta^D(x - y) \det g g^{\mu\nu}.$$

For proofs of these statements see page 93 of [this document](#).]

2. Maxwell's equations, quantumly.

- (a) Check that the oscillator algebra for the photon creation and annihilation operators

$$[\mathbf{a}_{ks}, \mathbf{a}_{k's}^\dagger] = \delta^3(k - k') \delta_{ss'}. \quad (1)$$

implies (using the mode expansion for $\mathbf{A}$) that

$$[\mathbf{A}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = -i\hbar \int d^3k e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} (\delta_{ij} - \hat{k}_i \hat{k}_j)$$

(and also $[\mathbf{A}_i(\vec{r}), \mathbf{A}_j(\vec{r}')] = 0$ and $[\mathbf{E}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = 0$).

Conclude that it's not possible to simultaneously measure $E_x(\vec{r})$ and $B_y(\vec{r})$.

- (b) Using the result of the previous part, check that the wave equation for $\mathbf{A}_i(x)$ follows from the Heisenberg equations of motion

$$\partial_t \vec{\mathbf{E}} = \frac{i}{\hbar} [\mathbf{H}, \vec{\mathbf{E}}].$$

3. **Goldstone boson.** Here is a simple example of the Goldstone phenomenon, which I mentioned briefly in lecture. Consider again the complex scalar field from the previous assignment.

Suppose the potential is

$$V(\Phi^*\Phi) = g(\Phi^*\Phi - v^2)^2$$

where g, v are constants. The important features of V are that (1) it is only a function of $|\Phi|^2 = \Phi\Phi^*$, so that it preserves the particle-number symmetry generated by the operator $\mathbf{q}$ which was the hero of a previous homework problem, and (2) $V(x)$ is not minimized at $x = 0$.

Treat the system classically. Write the action $S[\Phi, \Phi^*]$ in polar coordinates in field space:

$$\Phi(x, t) = \rho e^{i\theta}$$

where both ρ, θ are functions of space and time.

- (a) Consider constant field configurations, and show that minimizing the potential fixes ρ but not the phase θ .
- (b) Compute the mass² of the ρ field about its minimum, $m_\rho^2 = \frac{1}{2}\partial_\rho^2 V|_{\rho=v}$.
- (c) Now ignore the deviations of ρ from its minimum (it's heavy and slow and hard to excite), but continue to treat θ as a field. Plug the resulting expression

$$\Phi = v e^{i\theta(x,t)}$$

into the action. Show that θ is a massless scalar field.

- (d) How does the $U(1)$ symmetry generated by $\mathbf{q}$ act on θ ?

4. Angular momentum as Noether charge.

Consider an arbitrary local scalar field theory with Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ such that the action is invariant under rotations of space. A rotation acts on a scalar field by

$$\phi(x^i) \mapsto \phi(R_j^i x^j) \tag{2}$$

where R is a rotation matrix $R^T R = 1$. Consider an infinitesimal rotation $R_j^i = \delta_j^i + \omega_j^i$, for ω an arbitrary (small) real antisymmetric matrix specifying the plane and angle of rotation. Find the associated Noether currents in terms of ϕ .

Relate your answer to the form of the stress tensor. [Hint for interpreting the answer: a rotation is a space-dependent translation.]

If you prefer, you may consider instead the full Lorentz group, which acts by

$$\phi(x^\mu) \mapsto \phi(\Lambda^\mu_\nu x^\nu) \tag{3}$$

where Λ is a $D \times D$ matrix preserving the Minkowski metric $\eta_{\mu\nu}$.

Bonus problem: find the associated Noether charges in the case of a field theory of a vector such as electromagnetism, which transforms like $A_\mu(x) \mapsto (R^{-1})^\nu_\mu A_\nu(Rx)$.