University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 11

Due 11:59pm, Thursday December 12, 2024

In an effort to provide time to work on the final paper, the policy for the rest of the quarter is: all problems are bonus problems.

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1. **Representations of G_2.**

   In this problem we'll build representations of G_2 from scratch (*i.e.* from the Dynkin diagram). With enough effort we can see that G_2 is a subgroup of $\mathsf{SO}(7)$ and that it has an antisymmetric cubic invariant.

   (a) Check that the simple roots

   $$\alpha^1 = (0,1),\ \alpha^2 = (\sqrt{3}, -3)/2$$

   reproduce the Dynkin diagram of G_2.

   (b) Find the fundamental weights μ^a of G_2. Show that they are also roots! In particular you should find that $\mu^1 = 2\alpha^1 + \alpha^2, \mu^2 = 3\alpha^1 + 2\alpha^2$. This means that the root lattice and the weight lattice are the same in this case.

   (c) Find the orbit of μ^1 under Weyl reflections, and thereby draw the weight diagram for the representation with highest weight μ^1, $R_{(1,0)}$ (I recommend some symbolic software). Starting from the highest weight state, find a path connecting all these weights, where each step moves by (minus) a simple root. Conclude that $(0,0)$ must also be a weight vector. You should find that $R_{(1,0)} = \mathbf{7}$ is 7 dimensional. This is the fundamental representation of G_2 in the sense that all other reps appear in its tensor products.

   Bonus: label the weights by their $l^a - r^a$ vectors and check that the decompositions into $\mathsf{SU}(2)_{\alpha^a}$ multiplets makes sense.

   (d) Find the orbit of μ^2 under Weyl reflections and draw the weight diagram for the representation with highest weight μ^2, $R_{(0,1)}$. Conclude that $R_{(0,1)} = \mathbf{14}$ is the adjoint rep of G_2.

   (e) When we take tensor products, what happens to the weights? That is, given two reps $\mathbf{a}$ and $\mathbf{b}$ with highest weight vectors $\mu_{\mathbf{a}}$ and $\mu_{\mathbf{b}}$ respectively, what is the highest weight vector of $\mathbf{a} \otimes \mathbf{b}$?

   (f) What is the highest weight vector of $\Lambda^2 \mathbf{7}$? (Hint: the highest weight vector of $V \otimes V$ is symmetric under interchange of the two factors).

(g) [Bonus problem] Draw the weight diagram for $\Lambda^2\mathbf{7}$. Conclude that $\Lambda^2\mathbf{7} = \mathbf{7} \oplus \mathbf{14}$.

(h) [Bonus problem] Draw the weight diagram for $\text{Sym}^2\mathbf{7}$. Conclude that $\text{Sym}^2\mathbf{7}$ contains a copy of $R_{(2,0)}$, whose dimension we don't know yet. By counting the multiplicity of the $(0,0)$ weight vector, show that $\text{Sym}^2\mathbf{7} = R_{(2,0)} \oplus \mathbf{1}$. Conclude that $G_2 \subset \mathsf{SO}(7)$.

(i) [Bonus problem] Draw the weight diagram for $\Lambda^3\mathbf{7}$. Show that $\Lambda^3\mathbf{7} = R_{(2,0)} \oplus \mathbf{7} \oplus \mathbf{1}$. Conclude that G_2 has an antisymmetric cubic invariant. In fact G_2 can be defined as the subgroup of $\mathsf{SO}(7)$ which preserves an antisymmetric 3-index tensor.

[Cultural remark: this also means that it preserves a spinor of $\mathsf{SO}(7)$. For this reason, 7-manifolds with G_2 holonomy admit a covariantly-constant spinor (the generic orientable 7-manifold has holonomy $\mathsf{SO}(7)$). Compactification of supersymmetric field theories (such as 11-dimensional supergravity) on such manifolds therefore preserves some supersymmetry.]

(j) [Bonus problem] Show that the irrep with highest weight $a\mu_1 + b\mu_2$ with arbitrary $a, b \in \mathbb{Z}_{\geq 0}$ (*i.e.* the most general possible representation) is contained in the tensor product $\mathbf{7}^{\otimes n}$ for some n.

2. **$\mathsf{SO}(5)$ and $\mathsf{Sp}(4)$.**

(a) The simple roots of $\mathsf{so}(2n+1)$ are $e^i - e^{i+1}, i = 1..n-1, e^n$. Find the fundamental weights of $\mathsf{so}(5)$, μ^1 and μ^2. Build the weight diagrams for the two representations R_{μ^1} and R_{μ^2}.

(b) The simple roots of $\mathsf{sp}(2n)$ are $e^i - e^{i+1}, i = 1..n-1, 2e^n$. Find the fundamental weights of $\mathsf{sp}(4)$, μ^1 and μ^2 Build the weight diagrams for the two representations R_{μ^1} and R_{μ^2}.

(c) Compare.

(d) Argue that the anti-symmetric square of the spinor rep of $\mathsf{so}(4)$, $\Lambda^2\mathbf{4}$ contains a singlet.

3. **Spinor reps.**

(a) Find the constant $C(n)$ such that

$$\gamma_F \equiv C(n)\gamma_1 \cdots \gamma_{2n}$$

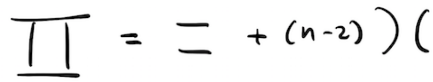
satisfies

$$\gamma_F = \gamma_F^\dagger \quad \text{and} \quad \gamma_F^2 = 1.$$

(Here γ_i are hermitian Majorana operators, satisfying $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$.)

(b) Check that $T_{i,2n+1} = -ST^{\star}_{i,2n+1}S^{-1}$ Conclude that the spinor rep of $\mathsf{SO}(2n+1)$ is not complex (where S is given in the lecture notes).

4. **Brain-warmer.** Verify the identity



(with $n = 3$) in $\mathsf{SO}(3)$ using birdtracks.

5. **Ramond-Ramond sectors.** [Bonus problem]

(a) The *Ramond* sector of the superstring worldsheet contains a Hilbert space on which 8 majorana operators γ_i act. The $\mathsf{SO}(8)$ acting on the index i in the fundamental is part of the spacetime symmetry. Physical states of the superstring are those which have definite eigenvalue of $\gamma_F = C\prod_{i=1}^{8}\gamma_i$ (where C is chosen so that $\gamma_F^\dagger = \gamma_F$ and $\gamma_F^2 = 1$).

The Ramond-Ramond sector of the closed superstring Hilbert space is the tensor product two Ramond sectors (one from right-moving modes and one from the left-moving modes on the closed string worldsheet). In type IIB, both copies have the same eigenvalue of γ_F and in type IIA, the two copies have opposite γ_F eigenvalue.

How do the physical states of each type of closed superstring transform under $\mathsf{SO}(8)$?

Removing the string theory jargon, the question is: how do $\mathbf{8}_+ \otimes \mathbf{8}_+$ and $\mathbf{8}_+ \otimes \mathbf{8}_-$ decompose into irreps of $\mathsf{SO}(8)$?

One way to do it is to consider the transformation law for objects of the form $\langle s_1 s_2 s_3 s_4 | \gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_k} | s_1 s_2 s_3 s_4\rangle$.

(b) [Super-bonus problem – requires some field theory] Consider the action

$$S[X^i, \psi^i] = \int d^2x ((\partial X)^2 + \psi \partial\!\!\!/ \psi),$$

where $i = 1..2n$. Take space to be periodic $x \equiv x + L$ and take periodic (Ramond) boundary conditions on the fermions fields (and the scalars). Show that its groundstates form a tensor product of two spinor representations of $\mathsf{SO}(2n)$.

Here is a hint: we can decompose the fields $X(x,t)$ and $\psi(x,t)$ into left-movers and right-movers:

$$X(x,t) = X_L(x-t) + X_R(x+t), \quad \psi(x,t) = \psi_L(x-t) + \psi_R(x+t).$$

If we impose periodic boundary conditions on both ψ_L and ψ_R we get the RR sector described above.

If we impose periodic boundary conditions on ψ_L and antiperiodic boundary conditions on ψ_R we get the R-NS sector. Show that these states transform as a spinor representation of $\mathsf{SO}(n)$ (and, consistent with the spin-statistics theorem, they are fermions).

6. **Schwinger bosons.**

   What happens if in our construction of spinor reps, we replace the fermions $\{c_a, c_b^\dagger\} = \delta_{ab}$ with bosons $[b_a, b_b^\dagger] = \delta_{ab}$?

   (a) Consider what representations this produces of the $\mathsf{SU}(n)$ subalgebra

   $$H_a = b_a^\dagger b_a - \frac{1}{2}, \quad E_{ab} = b_a^\dagger b_b, a \neq b.$$

   Hint: consider states of fixed particle number $\sum_a H_a = k$.

   First check that these operators actually do satisfy the $\mathsf{SU}(n)$ algebra.

   I recommend starting with the case of $n = 2$.

   (b) Can you make representations in this way of the full $\mathsf{SO}(2n)$ algebra which includes

   $$E'_{ab} = b_a^\dagger b_b^\dagger, a \neq b.$$

   What about $b_a^\dagger b_b^\dagger$ with $a = b$?

7. **Swap operator on qudits.** Let $\mathcal{H}_n = \text{span}\{|i\rangle, i = 1..n\}$ be an n-dimensional Hilbert space. In terms of the generators of $\mathsf{SU}(n)$ in the fundamental representation $(T_A)_{ij}$, write a formula for the *swap* operator SWAP on $\mathcal{H}_n \otimes \mathcal{H}_n$. The swap operator is defined by its action on the basis states:

   $$\mathsf{SWAP}\,|ij\rangle = |ji\rangle\,.$$

   Check that your answer makes sense for $n = 2$.

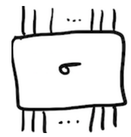
8. [Bonus problem] Compute the second Casimir for the adjoint rep of $\mathsf{SU}(n)$ using birdtracks.

   [Cultural remark: Notice that this quantity will appear in the vacuum polarization amplitude for the gluon, and hence the running of the QCD gauge coupling.]

   Using the result of the swap problem in the form above in the red circle, we have:

9. **Projector onto the antisymmetric tensor rep.**

For a positive integer k, consider the object (in the Brauer algebra)

$$P_{A^k} \equiv \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma}$$  .

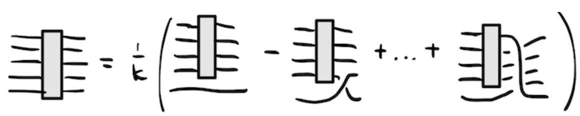
(a) Show that it is a projector.

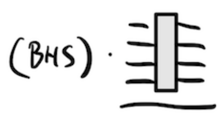
(b) Show that its trace is

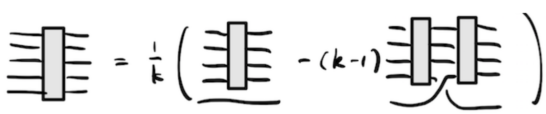
$$\mathrm{tr} P_{A^k} = \binom{n}{k} \equiv \frac{n(n-1)\cdots(n-k+1)}{k!}. \quad (1)$$

Hint: find a recursion relation between $\mathrm{tr}P_{A^k}$ and $\mathrm{tr}P_{A^{k-1}}$.

Elaboration of hint: First show that



Then by doing this:  show that



(c) Show that this vanishes if n is any integer less than k.

(d) [Bonus problem] Show that the expression (1) can be used as a generating function for the number of Young diagrams with n boxes and k columns.

10. **Schur-Weyl duality test.** [Bonus problem]

(a) Compute the character of the representation of S_3 on $\mathbf{n}^{\otimes 3}$ where $\mathbf{n}$ is the fundamental representation of $\mathsf{SU}(n)$.

(b) Decompose this representation into irreps of S_3. Check that the dimensions match the expected dimensions of the associated irreps of $\mathsf{SU}(n)$.

11. **Checking the reality of spinors.** [Bonus problem]

(a) Compute the characters of the spinor representations of $\mathsf{SO}(2n+1)$ and $\mathsf{SO}(2n)$ evaluated on an element of the Cartan subgroup $e^{\mathrm{i}\theta_a H_a}$.

(b) Verify the reality properties of the spinor reps from the behavior of the characters, by computing their Frobenius-Schur indicators or otherwise. In particular: what property of their character shows that the spinor irreps of $\mathsf{SO}(2n)$ with n odd are complex?

To do the integrals over the group, use the *Weyl integration formula*: for a class function $F(g)$,

$$\int_G dgF(g) = \int_T d\theta F(\theta)\Delta(\theta)\bar{\Delta}(\theta)$$

where T is the Cartan subgroup with coordinates $\theta_a, a = 1..r$, and

$$\Delta(\theta) \equiv \prod_{\text{positive roots},\alpha} \left(e^{\frac{\mathbf{i}\theta\cdot\alpha}{2}} - e^{-\frac{\mathbf{i}\theta\cdot\alpha}{2}}\right).$$

12. [Bonus problem]

(a) Show (using birdtracks or otherwise) that the character of the adjoint representation of $\mathsf{SU}(n)$ is

$$\chi_{\mathbf{adj}}(U) = \mathrm{tr}U\mathrm{tr}U^\dagger - 1.$$

(b) Conclude that the number of singlets in $\mathbf{adj}^{\otimes k}$ can be written as

$$\int_{\mathsf{SU}(n)} d\mu(U) \left(\mathrm{tr}U\mathrm{tr}U^\dagger - 1\right)^k$$

where $d\mu(U)$ is the Haar measure, with $\int_{\mathsf{SU}(n)} d\mu(U) = 1$.

(c) Using the large-n identity

$$\int_{\mathsf{SU}(n)} d\mu(U) \prod_\ell \left(\mathrm{tr}U^\ell\right)^{m_\ell} \left(\mathrm{tr}U^{\dagger\ell}\right)^{\bar{m}_\ell} \overset{n\gg1}{=} \prod_\ell \delta_{m_\ell\bar{m}_\ell} \ell^{m_\ell} (m_\ell)!\ , \qquad (2)$$

conclude that the number of singlets in $\mathbf{adj}^{\otimes k}$ is $\leq k!$.

(d) Understand why the identity (2) is true. Actually, at least for the case where $m_\ell = \delta_{\ell,1}$, this is possible using only information from the lecture notes! (No large-n assumption is required for this case.) Hint: use Schur-Weyl duality.