University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 10

Due 11:59pm, Thursday December 5, 2024

1. **Brain-warmers.**

   (a) If α^a, α^b are simple roots, show that $\alpha^a - \alpha^b$ cannot be a root.

   (b) Conclude using the previous result (and our analysis of $\mathsf{su}(2)$ subalgebras) that $A_{ab} \leq 0$ for $a \neq b$, where $A_{ab} \equiv \frac{2\alpha_a \cdot \alpha_b}{\alpha_a^2}$ is the Cartan matrix.

   (c) Check that Weyl reflection using the root α_a (call it w_a) acts on the root α_b by
   $$w_a(\alpha_b) = \alpha_b - A_{ba}\alpha_a \tag{1}$$
   where A_{ab} is the Cartan matrix.

   (d) [Bonus problem] Show that the Weyl reflections satisfy the relations
   $$w_a^2 = 1 \tag{2}$$
   and
   $$w_a w_b = w_b w_a \text{ if } A_{ab} = 0 \tag{3}$$
   and more generally
   $$(w_a w_b)^{m_{ab}} = 1 \tag{4}$$
   where
   $$m_{ab} = \begin{cases} 1 & \text{if } a = b \\ \frac{\pi}{\pi - \theta_{ab}} & \text{i } a \neq b \end{cases} \tag{5}$$
   where θ_{ab} is the angle between the simple roots α_a and α_b. That is, $m_{ab} = \{2, 3, 4, 6\}$ when the number of lines joining the a node to the b node in the Dynkin diagram is $0, 1, 2, 3$.

2. **Brain-warmer.** Consider the adjoint of $\mathsf{SU}(3)$ with highest weight state $|\mu^1 + \mu^2\rangle = |(1,1)\rangle$. Check that the two states with weight zero $|A\rangle \equiv E_{-\alpha^1}E_{-\alpha^2}|(1,1)\rangle$ and $|B\rangle \equiv E_{-\alpha^2}E_{-\alpha^1}|(1,1)\rangle$ are linearly independent (in agreement with the fact that there are two Cartan generators).

   Hint: show that two states $|A\rangle$ and $|B\rangle$ are linearly dependent only if $\langle A|A\rangle\langle B|B\rangle = \langle A|B\rangle\langle B|A\rangle$.

3. **Geometry problem related to the classification of Lie algebras.** [Bonus problem] Show that the sum of the three angles between three linearly independent vectors in $\mathbb{R}^3$ is less than 2π.