University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 7 – Solutions

Due 11:59pm Thursday, November 14, 2024

1. **Non-degeneracy of braiding in the 3d quantum double model.**

The title of this problem is a description of the physics situation where I encountered the following nice group theory fact. You don't need to know what it means to do the problem!

(a) Prove the following identity using character orthogonality: for any finite group G,

$$\sum_\alpha |Z_\alpha|^{n-2} = \sum_{a_1 \cdots a_n} N^1_{a_1 \cdots a_n} N^1_{\bar{a}_1 \cdots \bar{a}_n}$$

where Z_α is the centralizer of an element of conjugacy class α, $a_1 \cdots a_n$ are irreps of G, and $N^1_{a_1 \cdots a_n}$ is the number of times $R_{a_1} \otimes R_{a_2} \otimes \cdots R_{a_n}$ fuses to the identity[1]. The symbol $\sum_{a_1 \cdots a_n}$ means n independent sums over all the irreps of G.

[Cultural remark about the title of the problem, ignore if you want: The quantum double model is an exactly solvable lattice model with topological order. There is one for every finite group, and it can be put on any graph with the extra information of which faces (2-cells) are bounded by which edges of the graph. If we put this model on a 3d lattice, the LHS of this formula has an interpretation as the number of excitations shaped like an n-hole donut. The RHS has an interpretation as the number of collections of particles that can be braided non-trivially around such an object. The idea of "braiding non-degeneracy" in a system with topological order is that each topologically non-trivial excitation should be detectable from a distance by moving some *other* excitation around it.]

The solution has two steps. First we must remember that

$$N^1_{a_1 \cdots a_n} = \sum_\alpha n_\alpha \chi_{a_1}(\alpha) \chi_{a_2}(\alpha) \cdots \chi_{a_n}(\alpha).$$

[1] More precisely, $N^1_{a_1 \cdots a_n}$ is the dimension of the 'fusion space' $V^1_{a_1 \cdots a_n}$ in

$$R_{a_1} \otimes R_{a_2} \otimes \cdots R_{a_n} = \mathbf{1} \otimes V^1_{a_1 \cdots a_n} \oplus \cdots$$

where $\mathbf{1}$ is the trivial rep of G.

Then, for $n = 3$:

$$\sum_{abc} N^1_{abc} N^1_{\bar{a}\bar{b}\bar{c}} = \sum_{abc} |\frac{1}{|G|} \sum_\alpha n_\alpha \chi^\alpha_a \chi^\alpha_b \chi^\alpha_c|^2 \tag{1}$$

$$= \sum_{\alpha,\alpha'} \frac{n_\alpha n_{\alpha'}}{|G|^2} \sum_a \chi^\star_a(\alpha)\chi_a(\alpha') \sum_b \chi^\star_b(\alpha)\chi_b(\alpha') \sum_b \chi^\star_c(\alpha)\chi_c(\alpha') \tag{2}$$

$$= \sum_\alpha \frac{n_\alpha^2}{|G|^2} \left(\frac{|G|}{n_\alpha}\right)^3 = \sum_\alpha \frac{|G|}{n_\alpha} = \sum_\alpha |Z_\alpha|. \tag{3}$$

where I used row orthogonality three times,

$$\sum_a \chi^\star_a(\alpha)\chi_a(\alpha') = \delta_{\alpha\alpha'} \frac{|G|}{n_\alpha}.$$

More generally:

$$\sum_{a_1\cdots a_n} N^1_{a_1\cdots a_n} N^1_{\bar{a}_1\cdots\bar{a}_n} = \sum_{a_1\cdots a_n} |\frac{1}{|G|} \sum_\alpha n_\alpha \chi^\alpha_{a_1} \chi^\alpha_{a_2} \cdots \chi^\alpha_{a_n}|^2 \tag{4}$$

$$= \sum_{\alpha,\alpha'} \frac{n_\alpha n_{\alpha'}}{|G|^2} \left(\sum_a \chi^\star_a(\alpha)\chi_a(\alpha')\right)^n \tag{5}$$

$$= \sum_\alpha \frac{n_\alpha^2}{|G|^2} \left(\frac{|G|}{n_\alpha}\right)^n = \sum_\alpha \left(\frac{|G|}{n_\alpha}\right)^{n-2} = \sum_\alpha |Z_\alpha|^{n-2}. \tag{6}$$

(b) What is N^1_{abc} for the group S_3? Find a group where N^1_{abc} takes a value other than 0 or 1 for some choice of abc.

My solution is in this mathematica notebook. For S_3, S_4 all fusion multiplicities to the identity are 0 or 1. S_5, A_4, A_5 and the Frobenius group F_5 are examples of groups which have some reps with $N^1_{abc} > 1$.

Thanks to Bowen Shi and Jin-Long Huang for the collaboration which led to this problem. Here is the relevant paper, in case you are curious.

2. **Control-Z algebra.** [Bonus problem]

Recall that

$$\mathsf{CZ}_{ij} \equiv e^{\mathbf{i}\frac{\pi}{4}(1-Z_i)(1-Z_j)} = \mathsf{CZ}_{ji}$$

is the control-Z operation.

(a) Check that in the Z basis $(Z|s\rangle = s|s\rangle, s = 0,1)$ CZ acts as Z on the second bit only if the first bit is 1:

$$\mathsf{CZ}|0s\rangle = |0s\rangle, \mathsf{CZ}|1s\rangle = (-1)^s|1s\rangle, \quad \bar{0} \equiv 1, \bar{1} \equiv 0.$$

$\frac{1-Z}{2}$ is the projector onto $s=1$. So the exponent is only nonzero when both s_1 and s_2 are 1. When this is the case, we get $e^{\mathbf{i}\pi} = -1$.

(b) Check that

$$X_i \mathsf{CZ}_{ij} X_i = \mathsf{CZ}_{ij} Z_j. \tag{7}$$

Using $XZ = -ZX$, the LHS is

$$\begin{aligned} X_i e^{\mathbf{i}\frac{\pi}{4}(1-Z_i)(1-Z_j)} X_i &= X_i^2 e^{\mathbf{i}\frac{\pi}{4}(1+Z_i)(1-Z_j)} && (8)\\ &= e^{\mathbf{i}\frac{\pi}{4}(1+Z_i)(1-Z_j)} = e^{\mathbf{i}\frac{\pi}{4}((1-Z_i)(1-Z_j)+2Z_i(1-Z_j))} && (9)\\ &= \mathsf{CZ}_{ij} e^{\mathbf{i}\frac{\pi}{2}Z_i(1-Z_j)}. && (10)\end{aligned}$$

But $\frac{1-Z_j}{2}$ is the projector onto $Z_j = -1$, so $e^{\pm\mathbf{i}\frac{\pi}{2}(1-Z_j)} = Z_j$ for either sign.

3. **Practice at exponentiating matrices.**

(a) Check that the matrices $(J_i)^j{}_k = -\mathbf{i}\epsilon_{ijk}$ satisfy the $\mathsf{su}(2)$ algebra. These are the generators of the **3**-dimensional representation of $\mathsf{SU}(2)$.

One way is to plug them into mathematica. A better way is to use epsilon identities:

$$\begin{aligned} ([J_i, J_j])_{kl} &= (-\mathbf{i})^2 (\epsilon_{ikm}\epsilon_{jml} - \epsilon_{jkm}\epsilon_{iml}) && (11)\\ &= \delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - (\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}) && (12)\\ &= \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl} && (13)\\ &= \epsilon_{ijm}\epsilon_{mkl} && (14)\\ &= \mathbf{i}\epsilon_{ijm}(J_m)_{kl}. && (15)\end{aligned}$$

(b) What are the eigenvalues of $\hat{n} \cdot \vec{J}$, where $\hat{n}$ is a unit vector?

The eigenvalues of J_3 are the three possible values of m_z in the spin one rep, which are $-1, 0, 1$. One way to see this directly is to notice that $J_3 = \sigma^2 \oplus 1$. But

$$RJ_3R^{-1} = \hat{n} \cdot \vec{J},$$

for some rotation matrix R, so the RHS has the same eigenvalues as J_3.

(c) Find an explicit expression for $e^{\mathbf{i}\theta \hat{n}\cdot\vec{J}}$, where $\vec{J}$ is the vector of matrices above. Check that your answer is reasonable when $\hat{n} = \hat{z}$.

Hint: Show that $(\hat{n}\cdot\vec{J})^3 = \hat{n}\cdot\vec{J}$ and use this to sum the exponential series. One way to see this is to use the previous part to conclude that $(J^3)^2$ is a projector.

We can show directly that $P \equiv (\hat{n}\cdot\vec{J})^2$ is a projector:

$$(\hat{n}\cdot\vec{J})^2 = \hat{n}_i\hat{n}_j\frac{1}{2}\{J_i, J_j\}$$

so the kl component is

$$(\hat{n}\cdot\vec{J})^2_{kl} = \hat{n}_i\hat{n}_j\frac{1}{2}\left(\epsilon_{ikm}\epsilon_{jml} + \epsilon_{jkm}\epsilon_{iml}\right) = \hat{n}_i\hat{n}_j\frac{1}{2}\left(2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) = (\mathbb{I} - |\hat{n}\rangle\langle\hat{n}|)_{kl}$$

which is the projector onto the 2d subspace perp to $\hat{n}$. Alternatively, we can just appeal to the fact that we can change $\hat{n}$ by conjugation, and use the fact that we know that the eigenvalues of J^3 are $-1, 0, 1$, and so these must also be the eigenvalues of $\hat{n}\cdot\vec{J}$ for arbitrary $\hat{n}$, and so the eigenvalues of $(\hat{n}\cdot\vec{J})^2$ must be $1, 0, 1$, and it is hermitean, so it is a projector of rank two.

Alternatively, we could just notice that $(J_3)^3 = J_3$, and then conclude by rotation invariance that $(\hat{n}\cdot\vec{J})^3 = \hat{n}\cdot\vec{J}$.

A third method is to note that

$$0 = \prod_{\text{evals},\lambda_i}\left(\lambda_i - \hat{n}\cdot\vec{\mathcal{J}}\right) = \left(1 - \hat{n}\cdot\vec{\mathcal{J}}\right)\hat{n}\cdot\vec{\mathcal{J}}\left(-1 - \hat{n}\cdot\vec{\mathcal{J}}\right) = \hat{n}\cdot\vec{J} - \left(\hat{n}\cdot\vec{\mathcal{J}}\right)^3 . \tag{16}$$

Therefore

$$e^{\mathbf{i}\theta\hat{n}\cdot\vec{J}} = \sum_{k=0}^{\infty}\frac{(\mathbf{i}\theta)^k}{k!}(\hat{n}\cdot\vec{J})^k \tag{17}$$

$$= \mathbb{I} + P\left(-1 + \sum_{k\ \text{even}}\frac{(\mathbf{i}\theta)^k}{k!}\right) + (\hat{n}\cdot\vec{J})\sum_{k\ \text{odd}}\frac{(\mathbf{i}\theta)^k}{k!} \tag{18}$$

$$= \mathbb{I} + P\left(-1 + \cos\theta\right) + (\hat{n}\cdot\vec{J})\mathbf{i}\sin\theta \tag{19}$$

For $\hat{n} = \hat{z}$, this is the matrix

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1\end{pmatrix} + \begin{pmatrix} -1+\cos\theta & 0 & 0 \\ 0 & -1+\cos\theta & 0 \\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & \sin\theta & 0 \\ -\sin\theta & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1\end{pmatrix}$$

a rotation about the z-axis, by angle θ.

4. **Conjugacy classes of rotations.**

Consider a rotation matrix $O \in \mathsf{SO}(n)$. For purposes of this problem, take as given the fact that conjugation $O \to VOV^{-1}$ changes the axis of rotation, but not the angle.

(a) What are the eigenvalues of a rotation matrix?

An arbitrary 2d rotation is $O_2 = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$, where $c \equiv \cos\theta, s \equiv \sin\theta$ for some angle θ. Its eigenvalues are $e^{\pm \mathbf{i}\theta}$.

Consider the 3d rotation $O_3 = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $c \equiv \cos\theta, s \equiv \sin\theta$ for some angle θ. Any other rotation is related by conjugation $O_3 \to VO_3V^{-1}$, which doesn't change the set of eigenvalues. Its eigenvalues are $e^{\mathbf{i}\theta}, e^{-\mathbf{i}\theta}, 1$.

For $n > 3$, one possible generalization is $O_n = \begin{pmatrix} c & s & 0 & \cdots \\ -s & c & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \ddots \end{pmatrix}$, where $c \equiv \cos\theta, s \equiv \sin\theta$ for some angle θ. This is a rotation in the x_1x_2 plane; we can choose a different plane of rotation by conjugation $O \to VOV^{-1}$, which doesn't change the set of eigenvalues. Its eigenvalues are $e^{\mathbf{i}\theta}, e^{-\mathbf{i}\theta}, 1 \cdots 1$.

But actually this *isn't* the most general rotation matrix. In $n \geq 4$, we can independently rotate in two different planes, like

$$O_n = \begin{pmatrix} c & s & 0 & 0 & \cdots \\ -s & c & 0 & 0 & \cdots \\ 0 & 0 & c' & s' & \cdots \\ 0 & 0 & -s' & c' & \cdots \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix}. \tag{20}$$

This can't be made from the above expression by a conjuguation, but it is a rotation. The general rotation is obtained by conjugating this by some V. It has eigenvalues

$$e^{\mathbf{i}\theta}, e^{-\mathbf{i}\theta}, e^{\mathbf{i}\theta'}, e^{-\mathbf{i}\theta'} \ldots \tag{21}$$

For even n, thees come in pairs; for odd n there is an extra 1.

(b) What are the eigenvectors of a rotation matrix?

The eigenvectors of O_n in (20) are

$$\begin{pmatrix} 1 \\ \mathbf{i} \\ 0 \\ 0 \\ \cdots \end{pmatrix}, \begin{pmatrix} 1 \\ -\mathbf{i} \\ 0 \\ 0 \\ \cdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \mathbf{i} \\ \cdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\mathbf{i} \\ \cdots \end{pmatrix},$$

respectively. For odd n, there is an extra $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \cdots \end{pmatrix}$. For the general rotation, we must act on these by a rotation V.

(c) Given a rotation matrix, how can you determine the angle of rotation?

Compute the eigenvalues, take the log of the nontrivial ones, divide by $\mathbf{i}$.

But actually this is kind of a nonsense question: for $n \geq 4$ there can be many angles of rotation.

(d) Given a rotation matrix, how can you determine the planes of rotation?

The eigenvectors with the pairs of complex-conjugate eigenvalues span these planes.

Hint: solve this problem in two dimensions first. Then for general n use conjugation to put the matrix in a canonical form.

5. **Counting solutions to algebraic equations in a finite group.** [Bonus problem]

(a) [This part you did already on the previous homework] Starting from the definition of the Frobenius-Schur indicator for a representation R_a, find a formula for $\sigma(h)$, the number of solutions of the equation $g^2 = h$ in a finite group G. Hint: multiply the definition by $\chi_a^\star(h)$, sum over a, and use character orthogonality.

Show that the number of square roots $\sigma(h)$ is a class function.

I got this problem from Tony Zee's book.

We can rewrite the definition of the FS indicator as

$$\eta_a |G| = \sum_{h \in G} \sigma_h \chi_a(h) \tag{22}$$

where $\sigma_h \equiv$ the number of solutions of $g^2 = h$. Multiplying the BHS of (22) by $\chi_a^\star(h')$ and summing over irreps gives

$$\sum_a |G|\eta_a \chi_a^\star(h') = \sum_{h\in G} \sigma_h \chi_a(h)\chi_a^\star(h') = \sigma_{h'}|G|.$$

Therefore

$$\sigma_h = \sum_a \eta_a \chi_a^\star(h) = \sum_a \eta_a \chi_a(h)$$

where the last step follows since η_a is only nonzero for reps where $\chi_a(h) = \chi_a^\star(h)$.

The existence of a formula in terms of the character proves that σ is a class function.

(b) Redo the previous problem to count solutions of $g^3 = h$ in G. Write the answer in terms of the object

$$\eta_a^{(3)} \equiv \frac{1}{|G|}\sum_{g\in G} \chi_a(g^3).$$

What's the number of cube roots of (12) in S_3?

By the same calculation, $\eta^{(3)}$ determines the number of cube roots of an element $h \in G$ via

$$\sum_a \gamma_a \chi_a^\star(h).$$

For $S_3 = \langle a, b | a^3 = b^2 = e, ab = ba^2\rangle$ this gives (3,0,1) for the number of cube roots of elements of the three conjugacy classes $([e],[a],[b])$ (it is a class function). You can see that this is correct from this table:

conjugacy class	g	g^2	g^3
1	e	e	e
(123)	a	a^2	e
(123)	a^2	a	e
(12)	b	e	b
(12)	ba	e	ba
(12)	ba^2	e	ba^2

(c) [Super-bonus problem] Show that $\eta_a^{(3)}$ defined above is nonzero only when there is a cubic invariant of the representation R_a, that is $R_a \otimes R_a \otimes R_a = \mathbf{1} \oplus \cdots$. Does it say something more specific (*e.g.* about how many cubic invariants there are)? Is $\eta_a^{(3)}$ always an integer?

(d) Count homomorphisms of the trefoil knot group $\langle g, h | h^2 = g^3 \rangle$ to S_3. [Hint: $\frac{1}{|G|}\sum_g D^a(g^2)$ is an intertwiner.]

Suppose we want to count solutions of $k^2 g^3 = e$ (this the same as the relation above by a relabelling). Here's the idea: for an irrep a, $\frac{1}{|G|}\sum_g D^a(g^2)$ is an intertwiner (commutes with all the $D^a(h)$) and so Schur's lemma means it's proportional to $\mathbb{I}$. Taking trace of the BHS, we have

$$\frac{1}{|G|}\sum_g D^a(g^2) = \frac{\eta_a}{d_a}\mathbb{I}_a. \tag{23}$$

This is like un-tracing the FS indictator. Multiplying the BHS by $D^a(k^3)$, taking the trace and summing over k gives

$$\sum_{g,k}\chi^a(k^2 g^3) = \frac{|G|^2 \eta_a \eta_a^{(3)}}{d_a}.$$

If we let $\tau_h \equiv$ the number of solutions of $g^3 k^2 = h$, the LHS is $\sum_h \tau_h \chi^a(h)$. Multiplying the BHS by $\chi^a(h')^\star$ lets us solve for τ_h as above:

$$\tau_h = |G| \sum_a \frac{\eta_a \eta_a^{(3)}}{d_a}\chi_a(h)^\star.$$

Using the character table for S_3,

	S_3	n_C	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{2}$
$(1) =$	$\cdot$	1	1	1	2
$(12) =$	$\mathbb{Z}_2$	3	1	-1	0
$(123) =$	$\mathbb{Z}_3$	2	1	1	-1

this gives

conjugacy class	η_a	$\eta_a^{(3)}$
1	1	1
(12)	1	0
(123)	1	1

Compare the result $\eta_a^{(3)} = (1, 0, 1)_a$ to the following triple fusions:

$$\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1} = \mathbf{1}, \quad \mathbf{1}'\otimes\mathbf{1}'\otimes\mathbf{1}' = \mathbf{1}', \quad \mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = \mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{2}\oplus\mathbf{2}\oplus\mathbf{2}.$$

So $\mathbf{1}$ and $\mathbf{2}$ contain a singlet, but $\mathbf{1}'$ does not. Also, $\eta_a^{(3)}$ seems to be an integer.