University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 6 – Solutions

Due 11:59pm Thursday, November 7, 2024

1. **Fun with the Frobenius-Schur indicator.**

   (a) Compute the Frobenius-Schur indicator for the **2**-dimensional irrep of S_3 using the matrices you constructed on the homework 3 problem 6. If you get one, can you find the basis where it is real?

   We can find the explicit basis by remembering that the fundamental representation **3** of S_3 acts on the vertices of an equilateral triangle as (real) rotations and reflections, and $\mathbf{3} = \mathbf{2} + \mathbf{1}$. By starting with the matrices of the adjoint rep, and projecting them onto the subspace orthonormal to the singlet (which is the vector $(1, 1, 1)$), we get a set of real matrices describing the two-dimensional irrep of S_3.

   In fact, the resulting matrices are just the $2\pi/3$ rotations and reflections acting on a two-dimensional vector:

$$D(123) = e^{-\frac{2\pi}{3}\mathbf{i}\sigma^2} = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}, D(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (b) Consider the 2-dimensional representation of Q_8. Using the character table, decide if it is complex, real or pseudoreal.

   Using the fact that $(\pm 1)^2 = 1, (\pm\mathbf{i})^2 = (\pm\mathbf{j})^2 = (\pm\mathbf{k})^2 = -1$, the Frobenius-Schur indicator is

$$\eta_{\mathbf{2}} = \frac{1}{|Q_8|}\sum_{g\in Q_8}\chi_{\mathbf{2}}(g^2) = \frac{1}{8}\left(2\chi_{\mathbf{2}}(1) + 6\chi_{\mathbf{2}}(-1)\right) = \frac{1}{8}\left(2\cdot 2 + 6\cdot(-2)\right) = \frac{1}{8}\left(4 - 12\right) = -1.$$

   We conclude that the **2** is pseudoreal.

   Write down its representation matrices in terms of Pauli matrices and check your answer.

$$D(\pm 1) = \pm\mathbb{I}, D(\pm\mathbf{i}) = \pm\mathbf{i}X, D(\pm\mathbf{j}) = \pm\mathbf{i}Y, D(\pm\mathbf{k}) = \mp\mathbf{i}Z.$$

   Indeed there are complex numbers involved and we are not easily rid of them. It is a familiar fact that matrices satisfying the algebra of the three Pauli matrices (a Clifford algebra) can't be written without some **i**s in various places. To see this explicitly, notice that $D(g)^\star = YD(g)Y$, and the Pauli matrix Y is antisymmetric.

(c) Let $\sigma_h \equiv$ the number of square roots g of the element $h = g^2$ in the group G. Then by writing the Frobenius-Schur indicator for an irrep as

$$\eta_a \equiv \frac{1}{|G|} \sum_{g \in G} \chi_a(g^2) = \frac{1}{|G|} \sum_{h = g^2 \in G} \sigma_h \chi_a(h) \tag{1}$$

and using character orthogonality, show that

$$\sigma_h = \sum_{\text{irreps}, a} \eta_a \chi_a(h). \tag{2}$$

Multiplying (1) by $\chi_a^\star(g)$ and summing over a gives

$$\sum_a \chi_a^\star(g) \eta_a \overset{(1)}{=} \frac{1}{|G|} \sum_a \sum_h \sigma_h \chi_a(h) \chi_a^\star(g) = \sum_h \sigma_h \left(\frac{1}{|G|} \sum_a \chi_a(h) \chi_a^\star(g) \right)$$

This last expression in parens is only nonzero if h and g are in the same conjugacy class, in which case it is $\frac{|G|}{n_h}$ (where n_h is the number of elements in the conjugacy class of h). Since σ_h is only a property of the conjugacy class, the factor of n_h cancels out and we arrive at (2).

2. **Fun with Frobenius reciprocity.**

(a) Take each irrep of S_3 and decompose it into irreps of $\mathbb{Z}_3 \subset S_3$ (for example using the characters).

The character table for $\mathbb{Z}_3$ is $\chi_3 = \begin{matrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{matrix}$. The trivial rep gives the trivial rep. The sign rep of S_3 has $D(123) = 1 = D(321)$, so this also gives the trivial rep. The $\mathbf{2}$ has $D_{\mathbf{2}}(123) = \begin{pmatrix} \omega & \\ & \omega^{-1} \end{pmatrix}$ $D_{\mathbf{2}}(321) = \begin{pmatrix} \omega^{-1} & \\ & \omega \end{pmatrix}$, so its character as a rep of $\mathbb{Z}_3$ is $\chi_{\mathbf{2} \text{ as a rep of } \mathbb{Z}_3} = \begin{pmatrix} \text{tr}D(e) \\ \text{tr}D(123) \\ \text{tr}D(321) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$. Acting on this with $\chi_3^{-1} \chi_{\mathbf{2} \text{ as a rep of } \mathbb{Z}_3} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. So $\mathbf{2} = \mathbf{1}_1 \oplus \mathbf{1}_2$, where I label the rep of $\mathbb{Z}_n$ by $D_\ell(g^k) = \omega^{kl}$ as $\mathbf{1}_\ell$.

(b) For each of the irreps of $\mathbb{Z}_3$, construct the induced representation of S_3 (you did this already for the trivial representation) and decompose it into irreps of S_3 (for example using the characters).

Let's do them all at once. Choose as representatives of G/H the states $|e\rangle$ and $|(12)\rangle$, so we have $D(12)\,|e\rangle = |(12)\rangle$. For the rep that takes g^k to $\omega^{k\ell}$, $\ell = 0, 1, 2$, we have $D(g)\,|e\rangle = D(123)\,|e\rangle = \omega^{\ell}\,|e\rangle$ and $D(321)\,|e\rangle = D(g^2)\,|e\rangle = \omega^{-\ell}\,|e\rangle$. To find: $D(123)\,|(12)\rangle = D(123)D(12)\,|e\rangle = D(12)D(321)\,|e\rangle = D(12)\omega^{-\ell}\,|e\rangle = \omega^{-\ell}\,|(12)\rangle$. Similarly, $D(12)\,|(12)\rangle = D(12)^2\,|(12)\rangle = |e\rangle$. Therefore, the representatives of the two generators are

$$D(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D(123) = \begin{pmatrix} \omega^{\ell} & \\ & \omega^{-\ell} \end{pmatrix}, \quad \ell = 0, 1, 2.$$

The characters are $\chi_\ell \begin{pmatrix} e \\ (12) \\ (123) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ \omega^{\ell} + \omega^{-\ell} \end{pmatrix}$. For $\ell = 0$, we saw before that this gives $\chi^{-1}\chi_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, so the trivial rep of $\mathbb{Z}_3$ induces $\mathbf{1} \oplus \mathbf{1}'$. For $\ell = 1, 2$, we have $\chi^{-1} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, so these reps both induce the $\mathbf{2}$, whose matrices perhaps you recognize.

If our only goal were to decompose the result into irreps of G, we could use the formula

$$\chi_{\mathrm{Ind}_H^G[W]}(g) = \mathrm{Ind}_H^G[\chi_W](g).$$

to compute the characters of the induced reps from the characters of the irreps of H, without actually constructing the induced reps.

(c) Check Frobenius reciprocity.

We've found

irrep of S_3	$\overset{\text{Res}}{\to}$	irrep of $\mathbb{Z}_3$	irrep of $\mathbb{Z}_3$	$\overset{\text{Ind}}{\to}$	irrep of S_3
$\mathbf{1}$	$\to$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\to$	$\mathbf{1} \oplus \mathbf{1}'$
$\mathbf{1}'$	$\to$	$\mathbf{1}_0$	$\mathbf{1}_1$	$\to$	$\mathbf{2}$
$\mathbf{2}$	$\to$	$\mathbf{1}_1 \oplus \mathbf{1}_2$	$\mathbf{1}_2$	$\to$	$\mathbf{2}$

The number of times the $\mathbf{2}$ of S_3 appears in the rep induced from either $\mathbf{1}_1$ or $\mathbf{1}_2$ is 1. This is also the number of times they each appear in its decomposition. Similarly, the trivial rep of $\mathbb{Z}^3$ appears once in the decomposition of $\mathbf{1}'$ and of $\mathbf{1}'$, and hence it induces one of each. The same for each irrep of each group. It works!

(d) [Bonus problem] Verify Frobenius reciprocity for $A_4 \subset S_4$.

The character tables are

S_4		n_C	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3}'$
$(1) = \ydiagram{4}$	$\cdot$	1	1	1	2	3	3
$(12) = \ydiagram{3,1}$	$\mathbb{Z}_2$	6	1	-1	0	1	-1
$(12)(34) = \ydiagram{2,2}$	$\mathbb{Z}_2$	3	1	1	2	-1	-1
$(123) = \ydiagram{2,1,1}$	$\mathbb{Z}_3$	8	1	1	-1	0	0
$(1234) = \ydiagram{1,1,1,1}$	$\mathbb{Z}_4$	6	1	-1	0	-1	1

A_4		n_C	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{3}$
(1)	$\cdot$	1	1	1	1	3
(123)	$\mathbb{Z}_3$	4	1	ω	ω^2	0
(132)	$\mathbb{Z}_3$	4	1	ω^2	ω	0
$(12)(34)$	$\mathbb{Z}_2$	3	1	1	1	-1

Restriction is easy: obviously $\mathbf{1} \to \mathbf{1}$ and by definition of A_4, the sign rep of S_4, $\mathbf{1}'$, restricts to the trivial rep of A_4.

$$\chi_{\mathbf{2} \text{ of } S_4 \text{ as a rep of } A_4} = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} = \chi_{A_4} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

$$\chi_{\mathbf{3} \text{ of } S_4 \text{ as a rep of } A_4} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \chi_{A_4} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\chi_{\mathbf{3}' \text{ of } S_4 \text{ as a rep of } A_4} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \chi_{A_4} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\mathbf{3}' = \mathbf{3} \otimes \mathbf{1}'$, so they look the same to the A_4 subgroup.

Let's decompose the induced reps using the character formula: if R is a rep of H, then the character of its induced rep of G is

$$\chi_{\mathrm{Ind}(R)}(g) = \frac{1}{|H|} \sum_{k \in G} \chi_R(k^{-1} g k),$$

where $\chi_R(g) = 0$ if $g \notin H$. For example, $\chi_{\mathrm{Ind}(R)}(1234) = 0$ always, since no conjugation can change the cycle structure and turn this into an element of A_4. Similarly for $\chi_{\mathrm{Ind}(R)}(12) = 0$. For the elements which are in A_4, conjugating $(12)(34)$ by anything gives back $(12)(34)$ so the formula gives

$$\chi_{\mathrm{Ind}(R)}((12)(34)) = \frac{1}{12} 24 \chi_R((12)(34)) = 2\chi_R((12)(34)).$$

Finally, (123) can be conjugated to (321) by (12) or $(12)(34)$, so

$$\chi_{\mathrm{Ind}(R)}((123)) = \frac{1}{12}\left((1+8+8+3)\chi_R(123) + (6+6)\chi_R(321)\right) = \chi_R(123) + \chi_R(321),$$

just the average of the two characters of A_4. This gives

$$\chi_{\mathrm{Ind}(\mathbf{1})} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} = \chi_{S_4} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \chi_{\mathrm{Ind}(\mathbf{1}')} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} = \chi_{S_4} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\chi_{\mathrm{Ind}(\mathbf{1}'')} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} = \chi_{S_4} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \chi_{\mathrm{Ind}(\mathbf{3})} = \begin{pmatrix} 6 \\ 0 \\ -2 \\ 0 \\ 0 \end{pmatrix} = \chi_{S_4} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Therefore, altogether:

irrep of S_4	$\overset{\mathrm{Res}}{\to}$	irrep of A_4	irrep of A_4	$\overset{\mathrm{Ind}}{\to}$	irrep of S_4
$\mathbf{1}$	$\to$	$\mathbf{1}$	$\mathbf{1}$	$\to$	$\mathbf{1} \oplus \mathbf{1}'$
$\mathbf{1}'$	$\to$	$\mathbf{1}$	$\mathbf{1}'$	$\to$	$\mathbf{2}$
$\mathbf{2}$	$\to$	$\mathbf{1}' \oplus \mathbf{1}''$	$\mathbf{1}''$	$\to$	$\mathbf{2}$
$\mathbf{3}$	$\to$	$\mathbf{3}$	$\mathbf{3}$	$\to$	$\mathbf{3} \oplus \mathbf{3}'$
$\mathbf{3}'$	$\to$	$\mathbf{3}$			

which indeed satisfies Frobenius.

3. A_5 **is simple.**

(a) Show that an invariant subgroup $H \subset G$ is a union of conjugacy classes of G.

The definition of invariant subgroup is $ghg^{-1} \in H$ for all $g \in G, h \in H$. We know G is a union of its own conjugacy classes, and the ones that intersect H are completely in H.

(b) [Bonus problem] Find the number of elements of the conjugacy classes n_C of A_5. The answer is $n_C = 1, 15, 20, 12, 12$.

First decompose the 60 elements of A_5 into equivalence classes under conjugation by elements of S_5: this is just the even conjugacy classes of S_5: $e, (123), (12)(34), (12345)$ of sizes $1, 15, 20, 24$. When does a conjugacy class of S_5 split up into multiple classes of A_5? If not all elements with the same cycle structure are related by conjugation by an even element. So (123) splits as a class of A_3 because $(123) = g(321)g$, where $g = (12), (23)$ or

(13) for example, but no even permutation. If there were another odd permutation that commuted with one of these gs, then we could make an even permutation relating (123) and (321). But in S_5 there is such a permutation, namely (45). So (123) remains intact as a class of A_5. Not so for (12345) which splits in half.

By the way, here is a nice result for all n, which I learned from Eliot Kisiel: a class C_g of S_n splits into two classes of A_n of equal size if and only if its centralizer in S_n is entirely in A_n: $Z_g \subset A_n$. This follows by the orbit-stabilizer theorem: in S_n,

$$|C_g^{S_n}||Z_g| = |S_n|.$$

But under the hypothesis, the same equation holds in A_n:

$$|C_g^{A_n}||Z_g| = |A_n| = |S_n|/2.$$

Another way to see immediately that the 24-element class must split up is that 24 does not divide $|A_5| = 60$, but the size of the conjugacy class is the order of the stabilizer of any of its elements.

A nice point from Ruoyu Yin is that A_5 is realized as the rotations a regular dodecahedron. Elements of the form (12345) and (54321) are rotations by $2\pi/5$ and $8\pi/5$ about the centers of opposite pentagons. These cannot be conjugated to each other by any rotation.

(c) Show that A_5 is simple by checking for possible sums of n_Cs (necessarily including 1) that divide $5!/2$.

We can make 16, 21, 13, 25, 36... but none of these are divisors of 60. Therefore Lagrange forbids any invariant subgroup of A_5. I got this argument from Tony Zee's book.

4. **Induced representations.**

(a) Check that the definition of induced representation in the notes actually produces a representation of G in the sense that $D(g_1 g_2) |n, x\rangle = D(g_1)D(g_2) |n, x\rangle$ for all $g_{1,2} \in G$ and $|n, x\rangle \in W \times V_{G/H}$.

Recall that $D(g) |n, x\rangle = \sum_m |m, gx\rangle D^W(h)_{mn}$. Then on the one hand,

$$\begin{aligned} D(g_1)D(g_2) |n, x\rangle &= D(g_1) |m, g_2 x\rangle D^W(h_2)_{mn} \qquad g_2 a_x = a_{g_2 x} h_2 && (3) \\ &= D(g_1)D(a_{g_2 x}) |m, 0\rangle D^W(h_2)_{mn} && (4) \\ &= |\ell, g_1 g_2 x\rangle D^W(h_1)_{\ell m} D^W(h_2)_{mn} && (5) \\ &= |\ell, g_1 g_2 x\rangle D^W(h_1 h_2)_{\ell n} \qquad g_1 g_2 a_x = a_{g_1 g_2 x} h_1 h_2 \end{aligned}$$

On the other hand,

$$D(g_1g_2)\,|n,x\rangle = \sum_\ell |\ell, g_1g_2x\rangle\, D^W(h_{12})_{\ell n} \qquad g_1g_2a_x = a_{g_1g_2x}h_{12}$$

Comparing the expressions at right, we see that $h_{12} = h_1h_2$ and so the two states are the same.

(b) Show that the character of the induced representation $\mathrm{Ind}_H^G(W)$ is

$$\chi_{\mathrm{Ind}_H^G(W)}(g) = \mathrm{Ind}_H^G[\chi_W](g)$$

where the operation on the RHS is defined in the lecture notes.

$$\chi_{\mathrm{Ind}_H^G(W)}(g) \equiv \mathrm{tr}_{W\times V_{G/H}}(D(g)) = \sum_{n,x} \langle n,x|\,|D(g)\,|n,x\rangle \tag{6}$$

$$= \sum_{n,x} \langle n,x|\,\left(|m,gx\rangle\, D^W(h)_{mn}\right) \qquad ga_x = a_{gx}h \tag{7}$$

The inner product is $\langle n,x|m,gx\rangle = 0$ unless $gx = x$ in G/H, *i.e.* unless $ga_x = a_xh$. Then this is

$$\chi_{\mathrm{Ind}_H^G(W)}(g) = \sum_x \mathrm{tr}_W D^W(h)\delta_{ga_x=a_xh} \qquad \implies h = a_x^{-1}ga_x \tag{8}$$

$$= \sum_x \mathrm{tr}_W D^W(a_x^{-1}ga_x) = \frac{1}{|H|}\sum_{g_1\in G} \chi_W(g_1^{-1}gg_1) = \mathrm{Ind}_H^G[\chi_W](g). \tag{9}$$

(c) Prove Frobenius reciprocity:

$$\left\langle \psi, \mathrm{Res}_H^G[\phi]\right\rangle_H = \left\langle \mathrm{Ind}_H^G[\psi], \phi\right\rangle_G.$$

The LHS is

$$\left\langle \psi, \mathrm{Res}_H^G[\phi]\right\rangle_H = \frac{1}{|H|}\sum_{h\in H} \psi(h^{-1})\phi(h)$$

– the restriction map just forgets about the rest of G not in H. The RHS is

$$\left\langle \mathrm{Ind}_H^G[\psi], \phi\right\rangle_G = \frac{1}{|G|}\sum_{s\in G} \frac{1}{|H|}\sum_{s\in G} \psi(\underbrace{(g^{-1}sg)^{-1}}_{=g^{-1}s^{-1}g})\phi(s) \tag{10}$$

This first factor is zero unless $g^{-1}s^{-1}g \in H$, *i.e.* unless $\exists h \in H$ such that $s^{-1}g = gh$, that is, $s = gh^{-1}g^{-1}$. So we can rewrite it as

$$\text{RHS} = \frac{1}{|H|}\frac{1}{|G|}\sum_{s,g\in G}\sum_{h\in H}\delta_{h=g^{-1}s^{-1}g}\psi(h)\phi(s) \tag{11}$$

$$= \frac{1}{|G||H|}\sum_{h\in H, g\in G}\psi(h)\underbrace{\phi(gh^{-1}g^{-1})}_{=\phi(h^{-1})} \tag{12}$$

$$= \frac{1}{|H|}\sum_{h\in H}\phi(h^{-1}\psi(h) = \frac{1}{|H|}\sum_{h\in H}\psi(h^{-1})\phi(h) = \text{LHS} \tag{13}$$

where at the last step we relabelled the summation variable $h \to h^{-1}$.