University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 5

Comment: this problem set contains many problems, but most of them are simple and many of them are bonus problems.

Due 3:30pm Thursday, October 31, 2024

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1. **Brain-warmer.** Given a unitary representation D of a finite group G consider the objects

$$P_a \equiv \frac{d_a}{|G|} \sum_{g \in \mathsf{G}} \chi_a(g)^\star D(g) \tag{1}$$

for each irrep a of G.

   (a) Show that $P^\dagger = P$.

   (b) Show that $P_a P_b = \delta_{ab} P_a$.

2. **All functions on a group.** Show that the matrix elements of the unitary matrices representing the irreps of a finite group provide an orthonormal basis for all functions on the group (with respect to the inner product $\langle f_1 | f_2 \rangle = \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} f_1^\star(g) f_2(g)$).

3. **Diffusion on the vertices of the cube.**

   (a) The group $\mathsf{O} = S_4$ of rotational symmetries of the cube acts on functions on vertices of the cube. Decompose this representation into irreps of S_4.

   (b) Assign a temperature to each vertex of the cube T_i. Suppose the temperature evolves according to

$$T_i(t+1) = (1-\lambda) T_i(t) + \frac{\lambda}{3} \sum_{\text{nbrs} j \text{ of } i} T_j(t)$$

   (here $\lambda \in (0,1)$ measures the rate of diffusion). Find the rate at which the temperature function approaches its final distribution. (Do it without explicitly diagonalizing any matrices.)

   (c) [bonus problem] Construct explicit projectors onto the eigenmodes using the character table of S_4.

4. **Diffusion on the edges of the tetrahedron.** [bonus problem]

(a) Show that the group of rotations which are symmetries of a regular tetrahedron is isomorphic to A_4, the subgroup of S_4 consisting of just the even permutations.

(b) Construct the character table for A_4.

(c) A_4 acts on the *edges* of the tetrahedron in a 6-dimensional representation. Decompose this into irreps.

(d) Suppose we have a tight-binding model on the edges of the tetrahedron, $H = -t\sum_{\langle e_1 e_2\rangle} |e_1\rangle\langle e_2| + h.c.$ where two edges are regarded as neighbors if they both lie in the boundary of the same face. Find the spectrum.

5. **Projection operators.** [bonus problem] Write a program to make the pictures of the normal modes of the 'triatomic molecule'. Write it in such a way that it is easy to redo it for the generalization to n particles arranged in a regular n-sided polygon.

6. **Using the algebra of classes to construct the character table for S_3.** [bonus problem]

   (a) Write explicit expressions for the $\mathbf{C}_\alpha$ operators for the case $G = S_3$, and find the structure constants $c_{\alpha\beta}^{\gamma}$ of the algebra of classes.

   (b) Construct the matrices $(\mathbf{C}_\alpha)^{\gamma}{}_{\beta}$, check that they commute, and find their eigenvalues, λ_α^a

   (c) Check that you reproduce the character table for S_3 using the fact that the eigenvalues should be $\lambda_\alpha^a = \chi_\alpha^a/\chi_e^a$.