University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 4

Comment: this problem set contains many problems, but most of them are simple and many of them are bonus problems.

Due 3:30pm Thursday, October 24, 2024

1. **Brain-warmer.** If I tell you that $\langle \chi_R, \chi_R \rangle \equiv \frac{1}{|G|}\sum_{g\in G} \chi_R^\star(g)\chi_R(g) = 2$ for some representation R of a finite group G, do you know whether R is irreducible? Explain.

2. **Brain-warmer.** Given two representations A, B of a group G (with carrier spaces U and V respectively) and an intertwiner between them
$$\Lambda : U \to V, \quad \Lambda A(g) = B(g)\Lambda \forall g \in G$$
show that $\ker \Lambda \subset U$ and $\mathrm{Im} \Lambda \subset V$ are invariant subspaces. (Recall that this was the crucial ingredient in Schur's Lemma.)

3. **Character table for the quaternions.** Figure out the character table for the quaternion group Q_8 (on page 11 of the lecture notes) by whatever means necessary (don't look it up).

4. **Irreps and conjugacy classes.**

   Consider the object
$$S^a_\alpha \equiv \frac{1}{n_\alpha}\sum_{g\in C_\alpha} D^a(g),$$
   a linear operator on V_a, the carrier space for an irrep of G. C_α is a conjugacy class of G.

   (a) Show that S^a_α commutes with all the $D^a(g)$.

   (b) Use Schur's lemma to conclude that $S^a_\alpha = \lambda^a_\alpha \mathbb{I}_a$ and find λ^a_α in terms of familiar objects.

   (c) [Bonus problem] Conclude that $\mathbf{C}_\alpha \mathbf{P}^a = \lambda^a_\alpha \mathbf{P}^a$ where $\mathbf{P}^a = \frac{d_a}{|G|}\sum_{g\in G} (\chi^a(g))^\star \, \mathbf{g}$ is the projector in the group algebra associated with the irrep a and $\mathbf{C}_\alpha \equiv \frac{1}{n_\alpha}\sum_{g\in C_\alpha} \mathbf{g}$ are elements of the center of the group algebra.

5. **Statistics of cycle lengths of permutations.** [This is a bonus problem]

   (a) What fraction of permutations in S_n have just one big cycle?

(b) What fraction of permutations in S_{2n} have no cycle of length greater than n? What is the large-n limit of this number?

[Hint: Count the number of ways to assign elements to such a cycle.]

(c) What fraction of permutations in S_n have no fixed points? What is the large-n limit of this number?

[Hint: One way to do it is to use inclusion-exclusion. Another way is to find a recursion relation for $D(n)$, the number of elements of S_n with no fixed point. A third, trickier, way is to use our formula for cycle-lengths; to use this method, I had to pass through the grand canonical ensemble, *i.e.* sum over n.]

6. **Completely unrelated riddles.**

(a) There are 100 students in the group theory class. The professor is tired of grading and decides to let the students grade each others' exams. He arranges 100 boxes and uniformly randomly puts one exam in each box. The students must each open one box and grade the exam they find. What's the probability that no student grades their own exam?

(b) [I recommend this puzzle very highly.] The next year, there are again 100 students in the group theory class. This time, for the final exam, the evil professor decides to assign a group project. He arranges 100 numbered boxes and uniformly randomly puts the name of one student in each box. Each student is allowed to open half the boxes. If every student finds their own name then all the students pass, otherwise they all fail. The students are not allowed to communicate with each other after the box-opening begins, but they are allowed to develop a strategy together beforehand. What is a strategy that allows them all to pass with probability $> .3$?

(c) A group of n comic artists is playing a game of exquisite corpse. This means that each person draws one frame of the comic, and then passes it on to the next. If the order of artists is determined by a uniformly random element of S_n, what is the probability that the result is a comic with n frames, each by a different artist?