University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 3 – Solutions

Comment: this problem set contains many problems, but most of them are simple and many of them are bonus problems.

Due 3:30pm Thursday, October 17, 2024

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1. **Brain-warmer.**

(a) Consider the action of a group G on itself by left multiplication. What kinds of terms are allowed in the cycle indicator for this group action?

Compute the cycle indicators for the action by left multiplication of Q_8, of D_4 and of $\mathbb{Z}_n$ (do the case of n prime first, and do general n as a bonus problem). What do you get if you set all the $z_i = 1$?

By the Sudoku rule, all of the cycles in a given permutation for the right action of a group on itself must have the same length. Therefore the cycle indicator is a sum of terms of the form $z_j^{|G|/j}$ where j divides $|G|$.

For Q_8, it is $Z(Q_8, Q_8) = \frac{1}{8}\left(z_1^8 + z_2^4 + 6z_4^2\right)$ where the first term is the identity $(1)(-1)(\mathbf{i})\cdots$, the second is -1, which acts, in cycle notation, as $(1,-1)(\mathbf{i},-\mathbf{i})(\mathbf{j},-\mathbf{j})(\mathbf{k},-\mathbf{k})$, and then all of the other elements have two four-cycles. Explicitly:

Q_8	1	-1	$\mathbf{i}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{j}$	$\mathbf{k}$	$-\mathbf{k}$	cycle structure
1	1	-1	$\mathbf{i}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{j}$	$\mathbf{k}$	$-\mathbf{k}$	(1)(2)(3)(4)(5)(6)(7)(8)
-1	-1	1	$-\mathbf{i}$	$\mathbf{i}$	$-\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{k}$	(12)(34)(56)(78)
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{i}$	-1	1	$\mathbf{k}$	$-\mathbf{k}$	$-\mathbf{j}$	$\mathbf{j}$	(4231)(8765)
$-\mathbf{i}$	$-\mathbf{i}$	$\mathbf{i}$	1	-1	$-\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{j}$	(1324)(5768)
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{j}$	$-\mathbf{k}$	$\mathbf{k}$	-1	1	$\mathbf{i}$	$-\mathbf{i}$	()()
$-\mathbf{j}$	$-\mathbf{j}$	$\mathbf{j}$	$\mathbf{k}$	$-\mathbf{k}$	1	-1	$-\mathbf{i}$	$\mathbf{i}$	()()
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{k}$	$\mathbf{j}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{i}$	-1	1	()()
$-\mathbf{k}$	$-\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{i}$	1	-1	()()

For D_4, using the presentation $D_4 = \langle a, b | a^4 = e, b^2 = e, bab = a^3\rangle$, the multiplication table is

D_4	e	a	a^2	a^3	b	ab	a^2b	a^3b	cycle structure
e	e	a	a^2	a^3	b	ab	a^2b	a^3b	(1)(2)(3)(4)(5)(6)(7)(8)
a	a	a^2	a^3	e	ab	a^2b	a^3b	b	(1234)(5678)
a^2	a^2	a^3	e	a	a^2b	a^3b	b	ab	(13)(24)(57)(68)
a^3	a^3	e	a	a^2	a^3b	b	ab	a^2b	(4321)(8765)
b	b	a^3b	a^2b	ab	e	a^3	a^2	a	(15)(28)(37)(46)
ab	ab	b	a^3b	a^2b	a	e	a^3	a^2	(16)(25)(38)(47)
a^2b	a^2b	ab	b	a^3b	a^2	a	e	a^3	(17)(26)(35)(48)
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	e	(18)(27)(36)(45)

This gives $Z(D_4, D_4) = \frac{1}{8}\left(z_1^8 + 5z_2^4 + 2z_4^2\right)$. So D_4 and Q_8 are definitely different, even though they have the same character table.

For $\mathbb{Z}_n$, the answer depends a lot on the factors of n. If n is prime, it is $Z(\mathbb{Z}_p, \mathbb{Z}_p) = \frac{1}{p}\left(z_1^p + (p-1)z_p\right)$. If $n = pq$ where p, q are prime, it is $Z(\mathbb{Z}_{pq}, \mathbb{Z}_{pq}) = \frac{1}{pq}\left(z_1^{pq} + (p-1)z_p^q + (q-1)z_q^p + (q-1)(p-1)z_{pq}\right)$. A weak check that our answers are correct is that if we set $z_i = 1, \forall i$ we get $Z = 1$.

More generally, the cycle index is determined by the *order* n_g of each element g: $g^{n_g} = e$: $Z = \frac{1}{|G|}\sum_g z_{n_g}^{|G|/n_G}$. The order of an element $m \in \mathbb{Z}_n$ is $n/\gcd(m,n)$, where $\gcd(m,n)$ is the greatest common divisor. Therefore

$$Z = \frac{1}{n}\sum_{m=0}^{n-1} z_{n/\gcd(m,n)}^{\gcd(m,n)}.$$

This can also be written in terms of the Euler phi function $\varphi(k)$ = the number of integers less than k that are relatively prime to k:

$$Z = \frac{1}{n}\sum_{k|n} \varphi(k) z_k^{n/k}.$$

(b) Find the cycle index for the action of Q_8 on itself by conjugation.

$1, -1$ act trivially and therefore produce z_1^8 each. The other six elements each reverse the sign of *four* other elements upon conjugation (*e.g.* $\mathbf{ij}(-\mathbf{i}) = -\mathbf{j}$), while leaving the other four alone, and so gives $z_1^4 z_2^2$. For Q_8, I find

$$Z(Q_8, Q_8 \text{ by conjugation}) = \frac{1}{8}(2z_1^8 + 6z_1^4 z_2^2).$$

2. **Brain-warmer.** Given a group homomorphism $\phi : G \to K$, show that its kernel $H \equiv \ker\phi \equiv \{g \in G | \phi(g) = e \in K\} \subset G$ is always a subgroup. Further, show that it is a *normal* subgroup.

   From the definition, we have the (exact) sequence of group homomorphisms

   $$\{e\} \to H \xrightarrow{i} G \xrightarrow{\phi} K \to \{e\}$$

   with i is the inclusion map, and 'exact' means that the image of each map is the kernel of the next, *e.g.* $\ker\phi = \mathrm{Im}\, i \simeq H \equiv \{h \in G | \phi(h) = e\}$. H is a subgroup: $\phi(e) = e$, $e = \phi(h^{-1}h) = \phi(h^{-1})\phi(h)$, so every element of $\ker\phi$ has an inverse in $\ker\phi$.

   H is furthermore a normal subgroup because for any $g \in G, h \in H$, $\phi(g^{-1}hg) = \phi(g^{-1})\underbrace{\phi(h)}_{=e}\phi(g) = e$ is therefore in H.

   The sequence of maps above immediately says that $K = G/H$.

3. **Representation on cosets.** Given a group G and a subgroup H, we can construct a representation of G by its action on the cosets $x \in G/H$. The action is

   $$\begin{array}{ccc} G/H & \to & G/H \\ x = \{g_1, g_2 \cdots\} & \mapsto & \{gg_1, gg_2 \cdots\} \end{array}.$$

   To see more explicitly what this does, choose a representative x_i of each distinct coset. Then $G = \cup_i x_i H$, and for each $g \in G$, gx_iH is again a coset and therefore equal to one of the x_iH. So left multiplication by g permutes the cosets. [1]

   Consider the case where $H = \langle(123)\rangle$ is the subgroup of S_3 generated by the order-3 element (123). First decompose S_3 into cosets by H. Write out the matrices in a basis. What representation of S_3 do you get by the construction above? (If it is reducible, decompose it into irreps, for example by computing its character.)

   Let $b = (123), a = (12)$. The cosets are $eH = \{e, b, b^2\}, aH = \{a, ab, ab^2\}$. Multiplying by a on the left interchanges these two. Multiplying by b or b^2 does nothing, so:

   $$D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

[1] A word of clarification here: an action of a group G on a set (recall that for a set $X = \{x_1, ..x_k\}$ of order k, this is a homomorphism from $G \to S_k$) is also a representation in the following sense: define the carrier space $V = \mathrm{span}\{|x_i\rangle, i = 1..k\}$, and define the linear operators to act on the basis vectors by $D(g)|x\rangle = |gx\rangle$. This is a special kind of representation called a permutation representation, since the linear operators take basis vectors to basis vectors, rather than making linear combinations – the matrix elements in this basis are a single 1 in each row and column and zeros in all the other entries.

This 2d rep decomposes into $\mathbf{1} \oplus \mathbf{1}'$, spanned respectively by the symmetric and antisymmetric combinations of the two basis vectors.

To double-check this statement, we can compute its character, $\chi_{\text{this}} = \text{tr}D \begin{pmatrix} e \\ a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$. Acting on this with the inverse matrix of the character table, we get

$$\begin{pmatrix} m_{\mathbf{1}} \\ m_{\mathbf{1}'} \\ m_{\mathbf{2}} \end{pmatrix} = \chi^{-1} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

which says $\chi_{\text{this}} = \chi_{\mathbf{1}} + \chi_{\mathbf{1}'}$, in agreement with the claim above.

What about the case where $H' = \langle (12) \rangle$? In each case, decompose it into irreps.

The cosets are $eH' = \{e, a\}, bH' = \{b, ba\}, b^2H' = \{b^2, b^2a\}$. Using the relation $ab = b^2a$, we find with the basis specified by the above order of elements,

$$D(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This is just the defining representation $\mathbf{3} = \mathbf{1} + \mathbf{2}$ of S_3, as you can also check using the characters.

This construction plays an important role in the idea of induced representations. We will learn to call it the representation of G induced by the trivial representation of H.

4. **The class of inverses.** Given a conjugacy class c, define the class $\bar{c}$ to consist of the inverses of each of the elements in c. Convince yourself that this is well-defined. Show that for a unitary representation R,

$$\chi_R(\bar{c}) = \chi_R(c)^\star.$$

5. **Character exercise.** Recall the definition of the regular representation of a finite group G:

$$\mathcal{H}_{\mathsf{G}} \equiv \text{span}\{|g\rangle\,,\ g \in \mathsf{G}\}.$$

This space also carries a representation of $\mathsf{G} \times \mathsf{G}$ by

$$\Gamma(m, n)\,|h\rangle = \left|mhn^{-1}\right\rangle,\ (m, n) \in \mathsf{G} \times \mathsf{G}.$$

$\mathcal{H}_\mathsf{G}$ is also reducible as a representation of $\mathsf{G} \times \mathsf{G}$. Show that

$$\mathcal{H}_\mathsf{G} = \oplus_{R,\text{irreps of }\mathsf{G}} R \otimes \bar{R}$$

where $\bar{R}$ is the conjugate representation to R, with $\Gamma_{\bar{R}}(g) = \Gamma_R(g)^\star$.

Hint: Find the characters of $\mathcal{H}_\mathsf{G}$ as a representation of $\mathsf{G} \times \mathsf{G}$, and compute $\left\langle \chi_{\mathcal{H}_\mathsf{G}} | \chi_{R_i \otimes R_j} \right\rangle$, where R_i are all the irreps of G.

[I got this problem from Bowen Shi.]

A note about two notions of tensor product: In lecture we talked about taking two representations $R_{1,2}$ of a single group G and making the tensor product representation of G $R_1 \otimes R_2$. On the other hand, given a pair of groups, we can form the tensor product group, $G \times H$, and we can make a representation of $G \times H$ in the form $R^G \otimes R^H$ where $R^{G,H}$ are reps of G, H. The irreps of $G \times H$ are $R^G_a \otimes R^H_b$ where the factors are irreps of G, H. To see this, notice that the character table of $G \times H$ can be constructed as the tensor product of the character tables using this basis.

For $(m, n) \in \mathsf{G} \times \mathsf{G}$,

$$\chi_{\mathcal{H}_G}(m, n) = \text{tr}\Gamma(m, n) = \sum_{h \in \mathsf{G}} \langle h | \Gamma(m, n) | h \rangle = \sum_{h \in \mathsf{G}} \langle h | mhn^{-1} \rangle = \sum_{h \in \mathsf{G}} \delta_{h, mhn^{-1}}.$$

This last condition says that the character is only nonzero if $m = hnh^{-1}$ – m and n are conjugate. Using $\chi_{R_i \otimes R_j}(m, n) = \chi_{R_i}(m)\chi_{R_j}(n)$, the overlap is

$$\left\langle \chi_{\mathcal{H}_G} | \chi_{R_i \otimes R_j} \right\rangle = \frac{1}{|\mathsf{G}|^2} \sum_{(m,n) \in \mathsf{G} \times \mathsf{G}} \chi_{\mathcal{H}_\mathsf{G}}(m, n) \chi^\star_{R_i}(m) \chi^\star_{R_j}(n) \tag{1}$$

$$= \frac{1}{|\mathsf{G}|} \sum_{h \in \mathsf{G}} \frac{1}{|\mathsf{G}|} \sum_{m,n} \delta_{m, hnh^{-1}} \chi^\star_{R_i}(n) \chi^\star_{R_j}(m) \tag{2}$$

$$= \frac{1}{|\mathsf{G}|} \sum_{h \in \mathsf{G}} \frac{1}{|\mathsf{G}|} \sum_{m = hnh^{-1}} \chi^\star_{R_i}(n) \chi^\star_{R_j}(m) \tag{3}$$

$$= \underbrace{\frac{1}{|\mathsf{G}|} \sum_{h \in \mathsf{G}}}_{=1} \frac{1}{|\mathsf{G}|} \sum_{m} \chi^\star_{R_i}(m) \chi^\star_{R_j}(m) \tag{4}$$

$$= \delta_{R_i, \overline{R_j}}. \tag{5}$$

6. **Reps from characters.** Using the character table, make unitary representation matrices for the 2-dimensional irreducible representation of $D_3 = S_3$ in which $D(123)$ and $D(132)$ are diagonal. Note that $1 + \omega + \omega^2 = 0$ where $\omega \equiv e^{2\pi \mathbf{i}/3}$.

7. **Quotients of the spherical model.** [This is a bonus problem, since (a) it uses some perhaps-unfamiliar notions from statistical mechanics and (b) I made it up from scratch.]

   If you would like to see my (long and interesting) solution to this problem from HW2, please email me or submit a solution of your own.